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The extended Graetz problem for dipolar fluids

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Abstract

Thermally developing laminar flow of a dipolar fluid in a duct (pipe or channel) including axial conduction (Graetz problem extended) is investigated. The solutions are based on a self-adjoint formalism resulting from a decomposition of the convective diffusion equation for laminar flow into a pair of first-order partial differential equations. This approach, which is based on the solution method of Paputsakis et al. for a laminar pipe flow of a Newtonian fluid, is not plagued by any uncertainties arising from expansions in terms of eigenfunctions belonging to a non-self-adjoint operator. Then the eigenvalue problem is solved by means of the method of the weighted residual. Following this, the effect of the dipolar constant on the Nusselt number and temperature field are discussed in detail. Finally, it is shown that the Newtonian solution is a special case of the present result.
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1. Introduction

The theory of dipolar fluids is the simplest example of a class of non-Newtonian fluids called multipolar fluids. Bleustein and Green [1] studied the theory of dipolar fluids. Cowin [2] pointed out that dipolar fluids are a special case of fluids with deformable microstructure. Erdogan [3] has stated that this microstructure may consist of such entities as bubbles, atoms, particulate matter, ions or other suspended bodies. Straughan [4] has suggested that the theory of dipolar fluids should be capable of describing fluids made up of long molecules or (possibly) a suspension of long molecular particles. In a series of paper, Puri and Jordan [5–7], Jordan and Puri [8] have studied Stokes' first, second and unsteady Couette flow problem analytically for dipolar fluid respectively.

In a recent paper [9], Necas and Shilhavy examined the physical theory of multipolar fluids; where the theory is shown to be compatible with the principles of thermodynamics, as well as with the principles of material frame indifferences. In a follow up paper [10], Bellout et al. have explored some of the consequences of the theory formulated in [9]. The special flows examined in [10] relate to the simplest linear multipolar model which consists with the general formulation in [10], namely that of isothermal, incompressible, dipolar fluid; the specific constitutive relations are

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} - 2\mu_1 \Delta e_{ij} \tag{1}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k} \tag{2}$$

where τ_{ij} is the viscous tensor, τ_{ijk} the first multipolar stress tensor, p the pressure, Δ is the Laplacian operator,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{3}$$

the components of the rate of deformation tensor, corresponding to the velocity field v_i, $i = 1,2,3$ and the constant μ_0, $\mu_1 > 0$; which is required by consistency with the second law of thermodynamics in the form of the Clausius–Duhem inequality.

The analytical hydrodynamic solutions of the dipolar fluid pipe and channel flows were derived by Bleustein

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Nomenclature

$T(\xi,x)$	temperature
T_0	entrance temperature
T_1	final temperature
k_1	thermal conductivity
c	specific heat of fluid
l	dipolar constant
T_b	bulk temperature
Nu	Nusselt number
Pe	Peclet Number
$u(y)$ and $u(r)$	velocity

Greek symbols

ξ	coordinate
ρ	density
μ_0, μ_1	viscosity coefficient
Ω	axial energy
λ_j	eigenvalue

and Green [1], Erdogan and Gurgoze [11] and Bellout et al. [10] respectively.

In this report, thermally developing laminar flow (Graetz problem extended) of a dipolar fluid in a duct (pipe or channel) including axial conduction is analyzed and the effect of the dipolar constant on the temperature field and Nusselt number delineated. Literature review reveals that this thermally developing flow has not been studied before. In fact, the Graetz problem where the axial conduction is neglected has not been studied. This gives the motivation for the present work where approximate analytical solution is given for temperature field and heat conduction, which show the effect of the dipolar constant on the temperature distribution and heat conduction (Nusselt number), we also show that Newtonian solution is a special case of our work.

2. Formulation of the problem

The flow is assumed to be steady and laminar. The fluid properties are taken as constant and so no dependence of properties and model parameters on temperature will be considered. The boundary condition is that of an imposed constant temperature at the duct wall. Two flow geometries are considered, namely the plane case (channel flow) and the axisymmetric case (pipe flow), but the detail of derivation is given only pipe flow. At the end of the paper the result is given for channel flow as well.

It is also assumed that Fourier's law of heat convection is valid for the dipolar fluid [4]. Under these assumptions the hydrodynamic and the thermal problems are fully decoupled. The hydrodynamic solution is presented first, followed by the solution method and approximate analytical solution for the Graetz problem, which is the main focus of present report.

2.1. Hydrodynamic solution

The hydrodynamic solution for the pipe flow (Poiseuille flow, where velocity field has the form $\boldsymbol{v} = u(r)\boldsymbol{e}_z$) was derived by Bleustein and Green [1]. They assumed finite velocity along the center line ($r = 0$) of cylinder with $\tau_{r\alpha}(1) = M_\alpha$, $\alpha = r, \theta, z$, beside the boundary conditions $u(1) = u''(1) = 0$ and their solution for $M_z = 0$ can be written as

$$u(r) = 1 - r^2 + 2l^2 \frac{\left(I_0\left(\frac{1}{l}\right) - I_0\left(\frac{r}{l}\right)\right)}{\left(lI_1\left(\frac{1}{l}\right) - I_0\left(\frac{1}{l}\right)\right)} \quad (4)$$

where I_i, $i = 0, 1$ is the modified Bessel function of the first kind. For the plane case the velocity field has the form $\boldsymbol{v} = u(y)\boldsymbol{i}$; $(-1 \leqslant y \leqslant 1)$ and we have the analytical solution as

$$u(y) = 1 - y^2 - 2l^2 + 2l^2 \frac{\cosh\left(\frac{y}{l}\right)}{\cosh\left(\frac{1}{l}\right)} \quad (5)$$

where we used the dimensionless variables and $l^2 = \mu_1/\mu_0$ (dipolar constant).

2.2. Heat transfer problem

Fig. 1 shows the geometrical configuration and coordinate system. The characteristic length L denotes

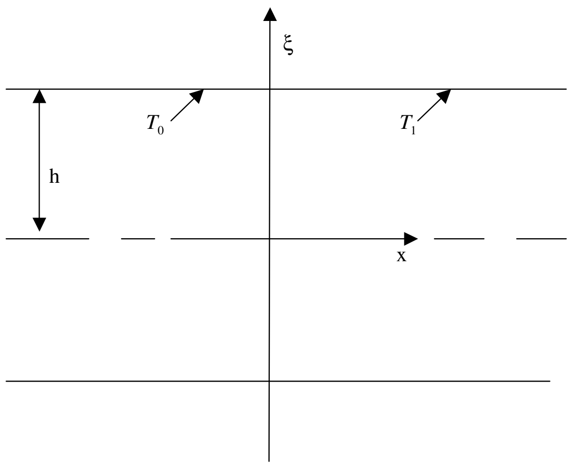

Fig. 1. Geometrical configuration and coordinate system.

half of the channel height h for the flow in a parallel plate channel or the radius R for the flow in a pipe. It is assumed that the laminar flow enters the duct with a hydrodynamically fully developed and with a uniform temperature profile T_0 for $x \to -\infty$. For $x \to \infty$ the flow attains the uniform temperature T_1. Under the assumption of constant fluid properties, the energy equation is given by

$$k_1 \frac{1}{r^k} \frac{\partial}{\partial \xi}\left(r^k \frac{\partial T}{\partial \xi} \right) = \rho c u \frac{\partial T}{\partial x} - k_1 \frac{\partial^2 T}{\partial x^2}, \tag{6}$$

where k_1, ρ and c stand for the thermal conductivity, density and specific heat of the fluid.

The boundary conditions for our problem are:

$$\xi(r \text{ for pipe flow}, y \text{ for planar flow}) = L: \quad T = T_0 \quad \text{for } x \leqslant 0 \quad \text{and} \quad T = T_1 \quad \text{for } x > 0,$$

$$\xi = 0: \quad \frac{\partial T}{\partial \xi} = 0, \quad \lim_{x \to -\infty} T = T_0, \quad \lim_{x \to \infty} T = T_1$$

and the index k which appears in Eq. (6) is equal to 0 and 1 for the planar case, and for the circular pipe case, respectively.

Next, we non-dimensionalise Eq. (6) by scaling lengths with characteristic length ($\xi^* = \xi/L$, $x^* = x/(LPe)$, $Pe = UL/k_1$), velocity with the average velocity ($u^* = u/U$) and temperature as

$$T^* = \frac{T - T_1}{T_0 - T_1}, \tag{7}$$

where T_0 represent the entrance temperature. Then, the resulting dimensionless energy equation is

$$\frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*} \right) + \frac{1}{Pe^2} \frac{\partial^2 T^*}{\partial x^{*2}} = u^* \frac{\partial T^*}{\partial x^*} \tag{8}$$

which expresses the so-called Graetz problem extended. The dimensionless boundary conditions are

$$\xi = 1: \quad T^* = 1, \ x \leqslant 0 \quad \text{and} \quad T^* = 0, \ x > 0;$$
$$\xi = 0: \quad \partial T^*/\partial \xi = 0 \quad \lim_{x \to = \infty} T^* = 1 \quad \text{and} \quad \lim_{x \to \infty} T^* = 0 \tag{9}$$

3. Method of solution

First, as discussed in detail in [12], it is easy to show that the partial differential Eq. (8) can be decomposed as (omitting the star)

$$\frac{\partial}{\partial x}\begin{bmatrix} T \\ \Omega \end{bmatrix} = \begin{bmatrix} \dfrac{Pe^2 u}{a_1(\xi)} & -\dfrac{Pe^2}{r^k a_1(\xi)} \dfrac{\partial}{\partial \xi} \\ r^k a_2(\xi) \frac{\partial}{\partial \xi} & 0 \end{bmatrix} \begin{bmatrix} T \\ \Omega \end{bmatrix} \quad \text{or}$$

$$\frac{\partial}{\partial x} \boldsymbol{F}(x, \xi) = \boldsymbol{L} \boldsymbol{F}(x, \xi) \tag{10}$$

where the function $\Omega(x, \xi)$ which can be called the axial energy flow through a cross sectional area of the height ξ is defined by

$$\Omega = \int_0^{\xi} \left[uT - \frac{1}{Pe^2} a_1(\xi) \frac{\partial T}{\partial x} \right] r^k \, \mathrm{d}\xi \tag{11}$$

Since the Eq. (8) is non-self-adjoint, this would give an incomplete set of eigenvalues for the solution. But, the remarkable feature of the operator $\boldsymbol{L}$ is that it gives rise to a self-adjoint operator even if the original equation is not self-adjoint. This is the result of inner product of two vectors where we define inner product of two vectors as

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \equiv \int_0^1 \left[\frac{a_1(\xi) r^k}{Pe^2} f_1(\xi) g_1(\xi) + \frac{1}{a_2(\xi) r^k} f_2(\xi) g_2(\xi) \right] \mathrm{d}\xi \tag{12}$$

and the domain for $\boldsymbol{L}$ is given by

$$D(\boldsymbol{L}) = \{ \boldsymbol{\Phi} \in H : \boldsymbol{L\Phi} \in H, \Phi_2(1) = \Phi_2(0) = 0 \} \tag{13}$$

Then it is easy to show that $\boldsymbol{L}$ is a symmetric operator in H, where H is an appropriate Hilbert space. For a detailed explanation see [12] or [13]. We note that the expression reduces to the pipe flow for $k = 1$, $\xi = r$, $a_1(\xi) = a_2(\xi) = 1$ and reduces to parallel plate channel for $k = 0$, $\xi = y$, $a_1(\xi) = a_2(\xi) = 1$. Thus a self-adjoint eigenvalue problem associated with the equation is given by

$$\boldsymbol{L\Phi}_j = \lambda_j \boldsymbol{\Phi}_j \tag{14}$$

where $\boldsymbol{\Phi}_j$ denotes the eigenvector corresponding to the eigenvalue λ_j. Expanding Eq. (14) and eliminating Φ_{j2}, we obtain

$$[r^k a_2(\xi) \Phi'_{j1}]' + r^k \left[\frac{\lambda_j a_1(\xi)}{Pe^2} - u \right] \lambda_j \Phi_{j1} = 0 \tag{15}$$

where we used $r^k a_2(n) \Phi'_{j1} = \lambda_j \Phi_{j2}$ and the equation is to be solved with the boundary conditions:

$$\Phi'_{j1}(0) = 0, \quad \Phi_{j1}(1) = 0 \tag{16}$$

and we have the normalizing condition

$$\Phi_{j1}(0) = 1 \tag{17}$$

which has been used for the eigenvectors. The solution of the equation possesses both positive and negative eigenvalues. This is because the operator $\boldsymbol{L}$ is neither positive nor positive definite. But all eigenvalues are real, since the operator is symmetric, and we can see from the Eq. (10), if $\lambda_j a_1(\xi)/Pe^2 \to 0$, the eigenvalue problem reduces to the parabolic classical Greatz problem.

We now consider the solution of Eq. (10). For this, we first take the inner product of both sides with $\boldsymbol{\Phi}_j$, which gives

$$\frac{\partial}{\partial x} \langle \boldsymbol{F}, \boldsymbol{\Phi}_j \rangle = \lambda_j \langle \boldsymbol{F}, \boldsymbol{\Phi}_j \rangle + h(x) \Phi_{j2}(1), \tag{18}$$

where $h(x)$ is defined as

$$h(x) = \begin{cases} 1, & x \leqslant 0 \\ 0, & x > 0 \end{cases} \tag{19}$$

and where we used the following equality

$$\langle \boldsymbol{LF}, \boldsymbol{\Phi}_j \rangle = \langle \boldsymbol{F}, \boldsymbol{L\Phi}_j \rangle + h(x)\Phi_{j2}(1) \tag{20}$$

which is a consequence of (12). Then Eq. (18) has the general solution

$$\langle \boldsymbol{F}, \boldsymbol{\Phi}_j \rangle = A_{\lambda_j} e^{\lambda_j x} + e^{\lambda_j x} \int^x e^{-\lambda_j s} \Phi_{j2}(1) h(s)\, ds, \tag{21}$$

where A_{λ_j} is the integration constant. Applying of the boundary condition (9) yields the solution

$$\langle \boldsymbol{F}, \boldsymbol{\Phi}_j \rangle = e^{\lambda_j^+ x} \int_{\infty}^x e^{-\lambda_j^+ s} \Phi_{j2}(1) h(s)\, ds + e^{\lambda_j^- x} \int_{-\infty}^x e^{-\lambda_j^- s} \Phi_{j2}(1) h(s)\, ds \tag{22}$$

Finally this expression is used in the eigenfunction expansion

$$\begin{bmatrix} T(\xi, x) \\ \Omega(x, \xi) \end{bmatrix} = \sum_j \frac{\langle \boldsymbol{F}, \boldsymbol{\Phi}_j \rangle}{\|\boldsymbol{\Phi}_j\|^2} \boldsymbol{\Phi}_j(\xi) \tag{23}$$

where the norm calculated from the Eq. (12). If we explicitly distinguish in Eq. (23) positive and negative eigenvalues, we obtain:

$$T(\xi, x) = 1 + \sum_{j=1}^{\infty} \frac{\Phi_{j2}^+(1)\Phi_{j1}^+(\xi)}{\lambda_j^+ \|\Phi_j^+\|^2} e^{(\lambda_j^+ x)} \quad \text{for } x \leqslant 0, \tag{24}$$

$$T(\xi, x) = -\sum_{j=1}^{\infty} \frac{\Phi_{j2}^-(1)\Phi_{j1}^-(\xi)}{\lambda_j^- \|\Phi_j^-\|^2} e^{(\lambda_j^- x)} \quad \text{for } x > 0 \tag{25}$$

Further the above equations can be written as (The reader is referred to Weigand et al. [13] for further details)

$$T(\xi, x) = 1 + \sum_{j=1}^{\infty} A_j^+ \Phi_{j1}^+(\xi) e^{(\lambda_j^+ x)} \quad \text{for } x \leqslant 0, \tag{26}$$

$$T(\xi, x) = \sum_{j=1}^{\infty} A_j^- \Phi_{j1}^-(\xi) e^{(\lambda_j^- x)} \quad \text{for } x > 0 \tag{27}$$

where the coefficient $A_j^{\pm}$ are given by

$$A_j^{\pm} = \left[\frac{\Phi_{j2}^{\pm}(1)\Phi_{j1}^{\pm}(\xi)}{\lambda_j^{\pm} \|\Phi_j^{\pm}\|^2} \right] \tag{28}$$

The bulk temperature T_b and the local Nusselt number Nu are then determined as

$$T_b = \int_0^1 u T r^k\, d\xi \bigg/ \int_0^1 u r^k\, d\xi, \tag{29}$$

$$Nu = -D\left(\frac{\partial T}{\partial \xi}\right)_{\xi=L} \bigg/ (T_b - T_1) \tag{30}$$

where D is the hydraulic diameter of the duct. Introducing the dimensionless variables given by Eq. (7) into the Eqs. (29) and (30) and using the temperature distributions (26) and (27), we obtain the following expressions for bulk temperature and Nusselt number and for the pipe flow:

$$\left. \begin{aligned} T_b^* &= 2 + \frac{\sum_{j=1}^{\infty} A_j^+ \left\{ \dfrac{\Phi_{j1}'^+(1)}{\lambda_j^+} + \dfrac{\lambda_j^+}{Pe^2} \displaystyle\int_0^1 \Phi_{j1}^+(r) r\, dr \right\} \exp(\lambda_j^+ x)}{B(l)}, \\ Nu &= -\frac{2\sum_{j=1}^{\infty} A_j^+ \Phi_{j1}'^+(1) \exp(\lambda_j^+ x)}{T_b^*} \\ &\text{for } x \leqslant 0 \end{aligned} \right\} \tag{31}$$

where $B(l)$ is found to be

$$B(l) = \frac{1}{2} + \frac{4l^2 I_0\left(\frac{1}{l}\right)}{l I_1\left(\frac{1}{l}\right) - I_0\left(\frac{1}{l}\right)} - \frac{4l^3 I_1\left(\frac{1}{l}\right)}{l I_1\left(\frac{1}{l}\right) - I_0\left(\frac{1}{l}\right)} \tag{32}$$

and

$$\left. \begin{aligned} T_b^* &= \frac{\sum_{j=1}^{\infty} A_j^- \left\{ \dfrac{\Phi_{j1}'^-(1)}{\lambda_j^-} + \dfrac{\lambda_j^-}{Pe^2} \displaystyle\int_0^1 \Phi_{j1}^-(r) r\, dr \right\} \exp(\lambda_j^- x)}{B(l)}, \\ Nu &= -\frac{2\sum_{j=1}^{\infty} A_j^- \Phi_{j1}'^-(1) \exp(\lambda_j^- x)}{T_b^*} \\ &\text{for } x > 0 \end{aligned} \right\} \tag{33}$$

4. Result and discussion

4.1. Numerical procedure and accuracy

We used the spectral Galerkin method to solve the eigenvalue problem (15)–(17). For this purpose, we used $\cos\left[\frac{1}{2}(2i-1)\pi\xi\right]$ as basis functions, which satisfy the boundary condition automatically. The resulting algebraic quadratic eigenvalue problem is solved to find the eigenvalues as well as the eigenvectors of the eigenvalue problem. In the case of thermally developing laminar flow of a Newtonian fluid in a pipe, we compared our results with that of the analytical one. We observed that our results are accurate up to 10 decimal place which give us confidence in the accuracy of our result for the dipolar fluid. In general, we found that a larger number of eigenmodes required with the decrease of Pe and $|x|$ as indicated in [14].

4.2. Results

For small values of dipolar constant, it is well known from the work of Bellout et al. [10] that fully

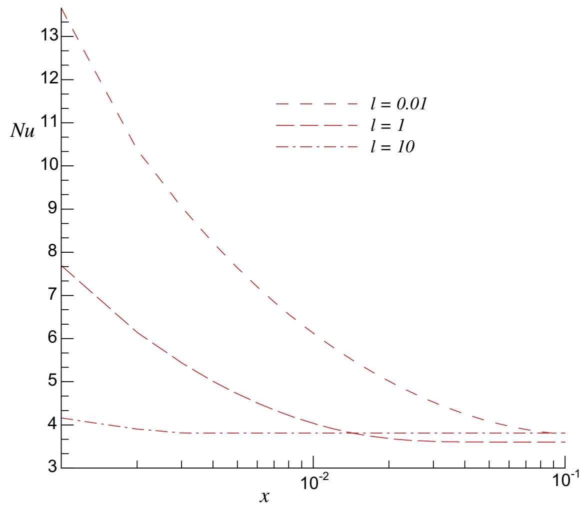

Fig. 2. The effect of the dipolar constants on the Nusselt number for $Pe = 1000$.

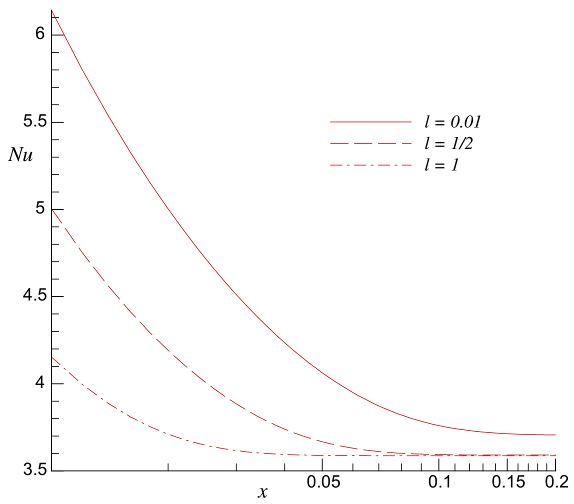

Fig. 4. The effect of the dipolar constants on the Nusselt number for $Pe = 100$.

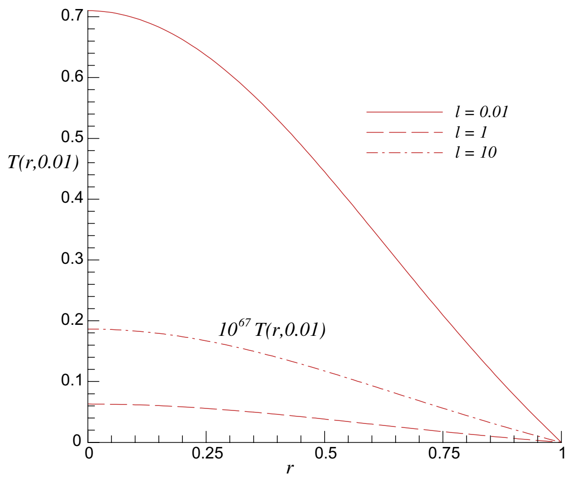

Fig. 3. The effect of the dipolar constants on the temperature field for $Pe = 1000$.

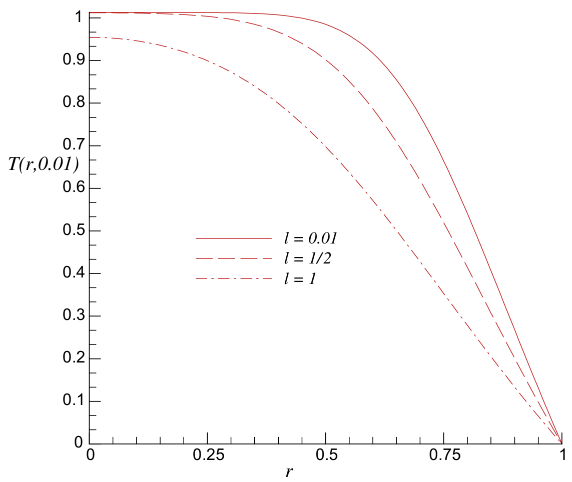

Fig. 5. The effect of the dipolar constants on the temperature field for $Pe = 100$.

developed velocity profile for dipolar fluid approaches the velocity profiles for Newtonian fluid. As expected, we found that the results for the Newtonian case ($l = 0$) were close to the results for small values of the dipolar constant l (this corresponds to the case $l = 0.01$ in Fig. 2). We also obtained the Graetz problem (no effect of axial conduction) for large enough Peclet number. To illustrate this case, we select Peclet number as 1000. Since there is no difference between the result for the Graetz problem (no effect of axial conduction) and $Pe = 1000$. We only examine the case of $Pe = 1000$ and this case is given in Fig. 2, which also shows the effect of the dipolar constant on the Nusselt number. But it is seen also from the Figs. 3 and 4, that, if we increase the value of the dipolar constant, we obtain the temperature distribution which is markedly different from that of the Newtonian case. This is shown in Figs. 3 and 4 respectively. It is seen from the Fig. 4 that dipolar constant $l > 1$ is not relevant for the applications (since it shows almost no temperature changes inside fluid). Hence, we consider the value of the dipolar constant between 0 and 1. If we decrease the value of the Peclet number, we expect the effect of

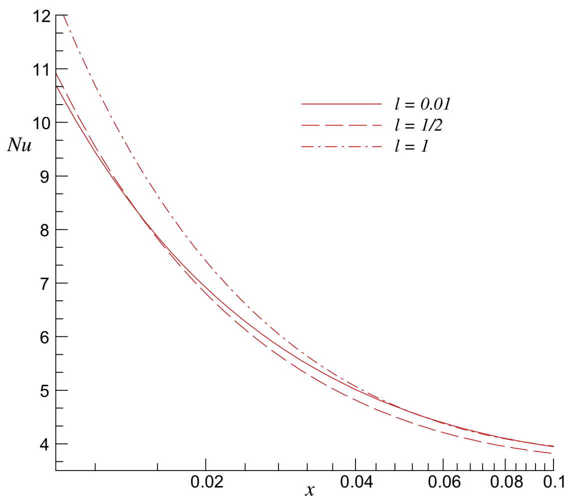

Fig. 6. The effect of the dipolar constants on the Nusselt number for $Pe = 10$.

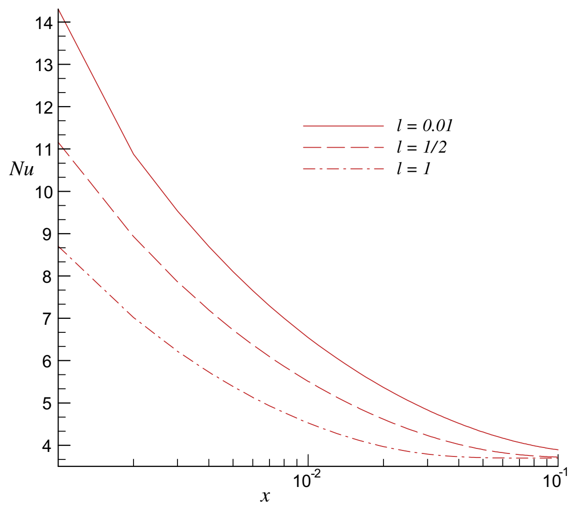

Fig. 8. The effect of the dipolar constants on the Nusselt number for the channel case.

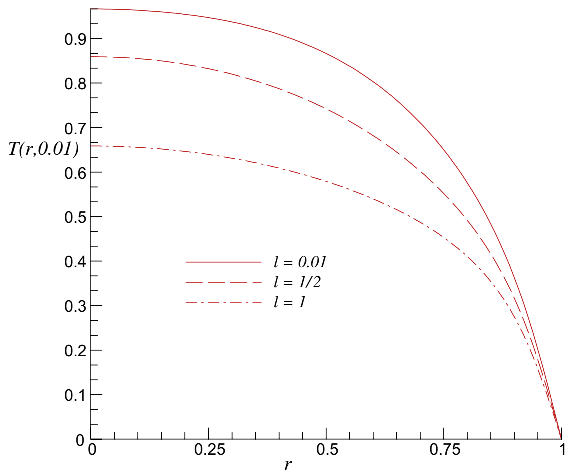

Fig. 7. The effect of the dipolar constants on the temperature field for $Pe = 10$.

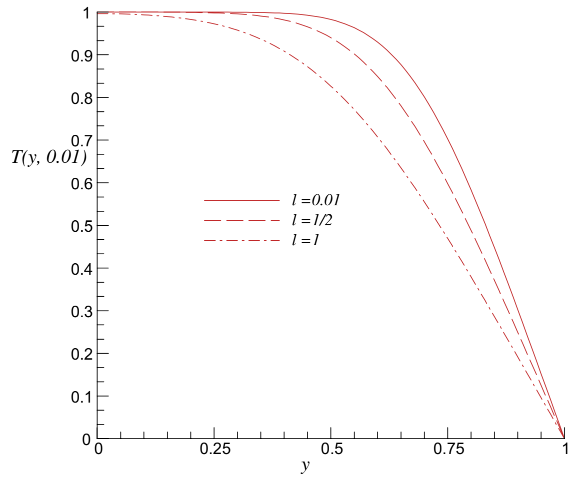

Fig. 9. The effect of the dipolar constants on the temperature field for the channel case.

the axial conduction on the temperature distribution as well as on the Nusselt number increase. This is shown in the Figs. 5–8. for $Pe = 100$ and 10 respectively. Again, the effect of the dipolar constant can be seen in these figures and we see that this effect can not be excluded. This also shows the relevance of the constitutive equation for dipolar fluids. Finally, the effects of the dipolar constant on the Graetz problem for the channel flow are shown in Fig. 9. We also note that we observed similar effect of the dipolar constant for the Graetz problem extended (including the axial conduction effect) for the channel case.

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